Pergamon

Int. J. Solids Structures Vol. 31, No. 5, pp. 733–744, 1994

0020–7683/94 $6.00 + .00

ON UNIQUENESS AND LOCALIZATION IN ELASTIC–DAMAGE MATERIALS

THIERRY DESOYER and FABRICE CORMERY
Laboratoire de Mécanique et de Physique des Matériaux, URA CNRS 863, ENSMA, 86034 POITIERS Cedex, France

(*Received* 29 *May* 1992; *in revised form* 6 *September* 1993)

Abstract—We consider isotropic elastic–damage behaviour such that the evolution law of the unique scalar internal variable characterizing material damage is associative ("d-associative" model). Two particular variants of modelling are considered. More precisely, and because of some analogies between these models and non-associative plasticity, we attempt to compare the loss of positiveness of second-order work and the localization criteria. These criteria are written in terms of critical damage. Thus, the damage value at loss of positiveness of second-order work is explicitly calculated in three-dimensional and plane strain cases. The procedure leading to the damage value at localization in the plane strain case is also presented. Both criteria are then compared for some loading paths. The results obtained indicate the localization occurring *before* loss of positiveness of the second-order work for some loading paths for one of two models. The same loading paths were tested with the other isotropic elastic–damage model; it is shown that localization always occurs after the loss of positiveness of second-order work (i.e. in softening phase). To endow this result with more generality, we then consider the problem of localization through a spectral analysis which finally shows that localization cannot take place in the hardening phase.

1. INTRODUCTION

The emergence of more or less large bands in which strain and damage become intense is frequently observed in structures undergoing inelastic strain and/or damage. Various physical mechanisms can initiate such a band: heterogeneities, thermal softening, ... Nevertheless, we will use here the term localization to refer to situations in which the concentration of strain and damage into a band emerges as an outcome of the very constitutive behaviour of the material in the context of a static boundary-value problem.

From this point of view, localization is generally considered as a particular loss of uniqueness of velocity field in the local rate problem. The question whether indeed the band formation can be preceded by other kinds of non-uniqueness must then be investigated. In particular, the comparison between the localization criterion and that of loss of positiveness of the second-order work is important. The sufficient conditions for uniqueness formulated by Hill (1958), in the context of a boundary value problem and for an associative flow rule or by Hueckel and Maier (1977) and Raniecki and Bruhns (1981) for non-associative flow rules are indeed never violated if the second-order work is positive.

Following Hadamard's studies on elastic stability (1903) extended to the inelastic context by Hill (1962) and Mandel (1966), Rice (1976) proposed to link the localization to the incipience of discontinuities of the velocity gradients. He showed that in the case of associative plasticity, the localization cannot occur in hardening phase ($\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} > 0$). In this study, we intend to compare both of the local criteria for loss of uniqueness in the specific context of elastic–damage behaviour.

The two models studied here are based on the assumption that the material damage is fully described by a single scalar internal variable d whose evolution is supposed associative (we will call these models "d-associative"). On the other hand, no particular hypothesis is formulated concerning the evolution of inelastic (i.e. damage related) strain $\dot{\boldsymbol{\varepsilon}}^{in}$. So, no particular configuration (associative-like for instance) is fixed between the stress and inelastic strain rate. The latter is solely governed by d and $\dot{d}$. This remark shows that even though these two models are "d-associative", they can resemble non-associative elastoplastic models admitting in particular $\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^{in} < 0$ at some points of the stress–strain path. The consequences of this peculiar form of constitutive behaviour on both of the local criteria previously mentioned must then be precisely examined.

In Section 2, we briefly present both elastic–damage models. The first of them is such that all the elastic moduli are modified in the same linear way by damage. On the other hand, only the shear modulus is modified by damage in the second model. The tangent matrix $\mathbf{L}$ (i.e. such that $\dot{\sigma} = \mathbf{L} : \dot{\varepsilon}$) is specified for both models to calculate both criteria.

The three-dimensional criterion of loss of positiveness of the second order work is explicitly calculated for each model in Section 3. In opposition to the elastoplastic case, this criterion is written here in terms of critical damage rather than in terms of critical hardening modulus. Furthermore this criterion is also specified for the plane strain case. The procedure leading to the localization criterion is then presented.

In Section 4, two local criteria are applied to some simple homogeneous plane strain test problems. Five proportional loading paths, characterized by the ratio $\beta = \dot{U}_y/\dot{U}_x = U_y/U_x$ are first tested. Finally, the results obtained allow us to compare both criteria and to determine if localization may occur in the "hardening" phase.

The few simple loading paths tested with the first elastic–damage model are such that localization always occurs after (or simultaneously with) the loss of positiveness of second-order work, i.e. in softening phase. To generalize this result, we then consider in Section 5 the localization by means of spectral analysis as suggested by Ottosen and Runesson (1991). This approach gives explicit analytical results for the critical damage value at localization which can be compared to the one corresponding to the second-order work criterion.

2. ELASTIC–DAMAGE MODELS

We present here two small strain damage models. The assumption is made that the material damage results in alterations of elastic moduli and is fully described by a single scalar internal variable d. This damage variable can be interpreted either as a relative porosity or as a microcrack density (assuming in this latter case that microcracks do not have peculiar orientations). The existence of a thermodynamical potential (here the free energy), ε-quadratic and d-linear, is then postulated:

$$\phi(\varepsilon, d) = \tfrac{1}{2}\mathbf{C}(d) : \varepsilon : \varepsilon, \tag{1}$$

where $\mathbf{C}(d)$ is the stiffness matrix of damaged material. Then, both the state relationships, are:

$$\sigma = \frac{\partial \phi(\varepsilon, d)}{\partial \varepsilon} = \mathbf{C}(d) : \varepsilon,$$

$$F_d = -\frac{\partial \phi(\varepsilon, d)}{\partial d} = -\tfrac{1}{2}\mathbf{C}' : \varepsilon : \varepsilon \quad \text{with } \mathbf{C}' = \partial \mathbf{C}(d)/\partial d. \tag{2}$$

In (2), F_d represents the thermodynamical force associated with d, i.e. the damage driving force; its physical meaning is a strain energy release rate due to damage for ε constant.

The relationships between elastic moduli and damage (i.e. $\mathbf{C}(d)$) may be obtained either by a homogenization process or by a phenomenological approach. We will consider here two particular forms of $\mathbf{C}(d)$; first of them (Model 1) is the henceforward classical phenomenological model, initially proposed by Lemaître and Chaboche (1978):

$$\mathbf{C}(d) = (1-d)\mathbf{C}^0, \tag{3}$$

where $\mathbf{C}^0$ is the stiffness matrix of the undamaged material. The model under consideration plays—due to its simplicity and, in the same time, due to its completeness—the analogous role in the mechanics of damage as does the Prandtl–Reuss model in the field of plasticity. Indeed, it is a sort of reference model and as such, it deserves that different aspects of bifurcation inherent to it be examined. This becomes even more crucial as the conclusions concerning various bifurcation events in the framework of elastic–plastic behaviour *cannot be* automatically extended to damage models. This feature will be developed further in this paper (see Section 5).

Model 1 is such that all the moduli are modified in the same way by damage; more particularly, the bulk modulus K decreases during damaging loading. This effect is widely refuted by experimental measurements for brittle (e.g. rock-like) materials. To yield this simple isotropic model roughly applicable to rock-like solids, Désoyer *et al.* (1990) proposed to modify it in such a way that only the shear modulus is altered by damage. Then, the stress–strain relation $(2)_1$ becomes (Model 2):

$$\boldsymbol{\sigma} = 2G(1-d)\boldsymbol{\varepsilon}' + K(\boldsymbol{\varepsilon}:\mathbf{1})\mathbf{1},$$
$$(\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} - \tfrac{1}{3}(\boldsymbol{\varepsilon}:\mathbf{1})\mathbf{1}), \tag{4}$$

where G is the shear modulus of undamaged material.

The damage criterion surface $g(F_d;\, d) = 0$ is written in the form:

$$g(F_d;d) = F_d - k(d) = 0 \quad \text{with } k(d) = \tfrac{1}{2}k_0(1+2\alpha d); k_0 > 0 \quad \text{and } \alpha \geqslant 0, \tag{5}$$

where $k(d)$ represents the actual damage threshold in the F_d-space with $k_0/2$ designating the initial one (at $d = 0$). The coefficient α describes the progressiveness of the damage limit. The greater α is, the more "ductile" is the stress–strain curve (see Fig. 1).

Assuming the damage rate $\dot{d}$ to be associative, we then have:

$$\dot{d} = \dot{\lambda}\frac{\partial g}{\partial F_d} = \dot{\lambda} \tag{6}$$

where $\dot{\lambda}$ is the "damage-multiplier" (by analogy with the elastoplastic terminology) which can be determined by using the consistency condition ($\dot{g} = 0$) yielding:

$$\dot{d} = \dot{\lambda} = \frac{[-\mathbf{C}':\boldsymbol{\varepsilon}:\dot{\boldsymbol{\varepsilon}}]^+}{k_0\alpha} \quad \text{where } \mathbf{C}' = \frac{\partial \mathbf{C}(d)}{\partial d}. \tag{7}$$

Remark

The function $k(d)$ is obtained from experiment and permits the expression of the dissipation D during damage loading ($D = F_d\dot{d} = k(d)\dot{d}$ since $g(F_d,\, d) = 0$ when $\dot{d} > 0$). Note that $k(d)$ plays, with respect to F_d, the role analogous as the hardening function in elastoplasticity in the stress space. $k(d)$ describes the dimension of the elastic domain referring to F_d while $k'(d)$ gives the evolution of this domain, like $k(p)$, (yield limit in classical isotropic hardening plasticity).

For each of these models, the tangent matrix $\mathbf{L}$ (such that $\dot{\boldsymbol{\sigma}} = \mathbf{L}:\, \dot{\boldsymbol{\varepsilon}}$) is given in somewhat general form by:

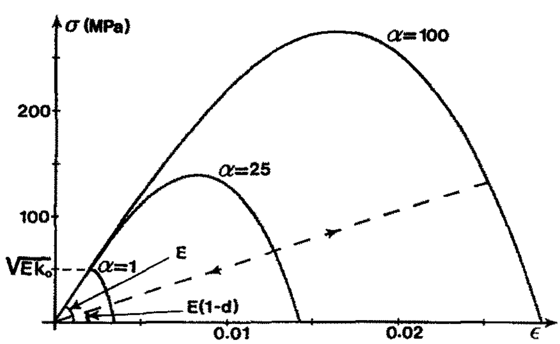

Fig. 1. Characteristic simple tension curve for elastic–damage behavior ($E = 25$ GPa; $\nu = 0.25$; $k_0 = 0.1$ MPa).

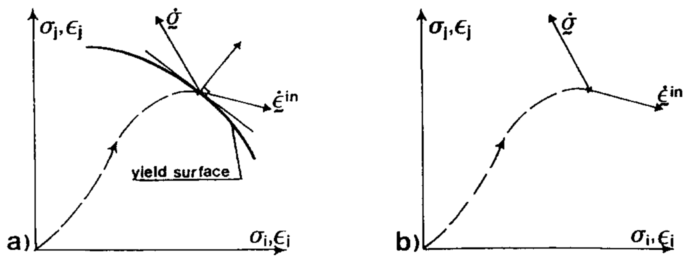

Fig. 2. Analogy possible between non-associative elastic–plastic behaviour and elastic–damage (d-associative) behaviour: configurations verifying $\dot{\boldsymbol{\sigma}}:\dot{\boldsymbol{\varepsilon}}^{in}<0$. (a) non-associative elastic–plastic behaviour; and (b) elastic–damage behaviour. Note that no plasticity-like criterion is specified for the damage related inelastic strain rate $\dot{\boldsymbol{\varepsilon}}^{in}$. The criterion introduced ($g \leqslant 0$) governs the damage evolution and influences $\dot{\boldsymbol{\varepsilon}}^{in}$ indirectly (through $\dot{d}$).

$$\mathbf{L}=\begin{cases}\mathbf{C}(d) & \text{if } g<0 \text{ or } g=0 \text{ and } \dot{g}<0,\\ \mathbf{H}=\mathbf{C}(d)-\dfrac{1}{k_0\alpha}(\mathbf{C}':\boldsymbol{\varepsilon})\otimes(\mathbf{C}':\boldsymbol{\varepsilon}) & \text{if } g=0 \text{ and } \dot{g}=0.\end{cases} \tag{8}$$

Note that the difference between the two models resides in the form of $\mathbf{C}'$.

By way of conclusion of this brief account on both elastic–damage models, note that no particular assumptions are made on the evolution law of the inelastic part of the strain rate tensor due to damage $\dot{\boldsymbol{\varepsilon}}^{in}$. It can be written as:

$$\dot{\boldsymbol{\varepsilon}}^{in}=\dot{d}(\mathbf{C}^{-1}(d))':\mathbf{C}(d):\boldsymbol{\varepsilon} \quad \text{with } (\mathbf{C}^{-1}(d))'=\frac{\partial \mathbf{C}^{-1}(d)}{\partial d}. \tag{9}$$

In fact, $\dot{\boldsymbol{\varepsilon}}^{in}$ may eventually admit the configuration illustrated in Fig. 2 usually characterizing non-associative elastic–plastic behaviour ($\dot{\boldsymbol{\sigma}}:\dot{\boldsymbol{\varepsilon}}^{in}<0$). So, referring to the results given by Rice and Rudnicki (1979) and Borré and Maier (1989) we can neither assure that the continuous bifurcation (i.e. $\mathbf{L}=\mathbf{H}$ inside and outside the band) occurs before the discontinuous one (i.e. $\mathbf{L}=\mathbf{H}$ inside the band and $\mathbf{L}=\mathbf{C}(d)$ outside the band) nor that localization happens only in the softening phase.

3. NON-UNIQUENESS CRITERIA

3.1. *Localization criterion*

Following studies by Rice (1976) and Rice and Rudnicki (1980) on localization and more recent works by Chambon (1986) and Benallal *et al.* (1989), we choose for the localization criterion the loss of ellipticity of continuous equilibrium rate relationships (we do not consider here the effects of free boundaries and interfaces on localization).

Let us consider a macroscopically homogeneous material element subject to a homogeneous stress or displacement at its boundaries. At some loading range, it deforms in a homogeneous manner (i.e. stress, strain and more generally all internal variable fields are homogeneous). For some loading level, the bifurcation of the rate of displacement $\dot{u}_{i,j}$ may occur across a fixed singular surface (localized band) leading to damage localization as well. Considering the local rate problem, the localization implies the difference between the value of $\dot{u}_{i,j}$ for the bifurcated and primary fields; the corresponding strain rate $\dot{\boldsymbol{\varepsilon}}$ becomes discontinuous across the band of a local normal vector $\mathbf{n}$, i.e.:

$$[\dot{\boldsymbol{\varepsilon}}]=(\mathbf{g}\otimes\mathbf{n})_s, \tag{10}$$

where $\mathbf{g}$ is the amplitude of strain rate jump.

Denoting by $\dot{\boldsymbol{\sigma}}^1$ the stress rate within the band and by $\dot{\boldsymbol{\sigma}}^0$ the stress rate outside the band, conservation of equilibrium across the band requires:

$$\dot{\boldsymbol{\sigma}}^1 \cdot \mathbf{n} = \dot{\boldsymbol{\sigma}}^0 \cdot \mathbf{n}. \tag{11}$$

Accounting for strain rate jump and tangent matrix definitions, this condition can be expressed as:

$$\mathbf{L}^1 : (\dot{\boldsymbol{\varepsilon}}^0 + [\dot{\boldsymbol{\varepsilon}}]) \cdot \mathbf{n} = \mathbf{L}^0 : \dot{\boldsymbol{\varepsilon}}^0 \cdot \mathbf{n}, \tag{12}$$

where $\dot{\boldsymbol{\varepsilon}}^0$ is the strain rate outside the band. Two different cases must then be examined:

(i) $\mathbf{L}^1 = \mathbf{L}^0 = \mathbf{H}$ ("continuous bifurcation": damage-loading inside and outside the band).

In this case, (12) becomes:

$$\mathbf{H} : [\dot{\boldsymbol{\varepsilon}}] \cdot \mathbf{n} = 0. \tag{13}$$

Accounting for (10) and minor symmetries of $\mathbf{H}(H_{ijkl} = H_{ijlk})$, (13) gives:

$$(\mathbf{n} \cdot \mathbf{H} \cdot \mathbf{n}) \cdot \mathbf{g} = 0. \tag{14}$$

Finally, a necessary and sufficient condition for the incipience of a strain rate jump across a locally planar band oriented by $\mathbf{n}$ is given by:

$$\det(\mathbf{n} \cdot \mathbf{H} \cdot \mathbf{n}) = 0. \tag{15}$$

(ii) $\mathbf{L}^1 = \mathbf{H}$ and $\mathbf{L}^0 = \mathbf{C}(d)$ ("discontinuous" bifurcation)

In such a case, which corresponds to a damage loading inside the band and an elastic unloading outside, Borre and Maier (1989) have shown that a necessary and sufficient condition for the incipience of localization can be written as:

$$\det(\mathbf{n} \cdot \mathbf{H} \cdot \mathbf{n}) \leqslant 0. \tag{16}$$

Note that, owing to the fact that this inequality is not strict, continuous and discontinuous bifurcation may occur simultaneously. Nevertheless, we will later restrict our consideration to continuous bifurcation condition of localization. In the case of plane strain in the (**x**, **y**) plane, for both of the elastic damage models under consideration, (15) is equivalent to:

$$A_1 \cos^4\theta + A_2 \cos^2\theta + A_3 \sin\theta \cos 3\theta + A_4 \sin\theta\cos\theta + A_5 = 0,$$
$$(\cos\theta = \mathbf{n} \cdot \mathbf{y}), \tag{17}$$

where the coefficients $A_i (i = 1\text{–}5)$ are functions of material constant (E, ν; k_0, α) and of $\boldsymbol{\varepsilon}$ and d (see Appendix 1 for more details). Thus, for a given mechanical state ($\boldsymbol{\varepsilon}$, d), the eventual solution of (15) gives the orientations of localization bands. The critical damage value at localization is denoted as d_ℓ.

3.2. *Loss of positiveness of the second-order work criterion*

The choice of positiveness of second-order work as the uniqueness criterion is motivated by the fact that the sufficient conditions for uniqueness, established by Hill (1958) for the associative flow rule or by Hueckel and Maier (1977) and Raniecki and Bruhns (1981) for the non-associative flow rule in the context of a boundary value problem, are never violated if the second-order work is positive. It is written:

$$W = \tfrac{1}{2}\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} \geqslant 0 \quad \forall \dot{\boldsymbol{\varepsilon}}. \tag{18}$$

Thus, the loss of positiveness of second-order work is equivalent to the requirement:

$$\exists\, \dot{\boldsymbol{\varepsilon}} \,|\, W = \tfrac{1}{2}\dot{\boldsymbol{\varepsilon}} : \mathbf{L} : \dot{\boldsymbol{\varepsilon}} = 0, \tag{19}$$

which constitutes what we will call later on, for the sake of simplicity, the "second-order work criterion". The critical damage d_s solution of (19) is found to correspond to the solution of a constrained minimization problem (see Appendix 2). In the three-dimensional case, it is given for both the elastic–damage models by the single formula as follows:

$$d_s = \frac{\alpha - 1}{3\alpha}. \tag{20}$$

Note that d_s is in some manner a sort of material characteristic, fully independent of loading path.

Remarks

(i) When $\alpha < 1$, (20) leads to an inadmissible value of $d_s (d_s < 0)$. In fact, and in opposition to what is assumed to prove (19) (see Appendix 2), such materials cannot have any mechanical state ($\boldsymbol{\varepsilon}$, d) with $d \neq 0$ and $\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} > 0$. They are immediately softening when the initial damage limit is reached.

(ii) The plane strain case imposes an additional condition to the minimization problem leading to the second-order work criterion. It does not however modify the previous criterion for the first elastic–damage model studied (Model 1). Nevertheless, for Model 2 this criterion becomes:

$$\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - \frac{9K + 2G(1-d)}{18K + 6G(1-d)} (\boldsymbol{\varepsilon} : 1)^2 = \frac{k_0 \alpha}{2G} (1-d). \tag{21}$$

Hence, in opposition to the three-dimensional case, this criterion is here a function of the loading path.

4. COMPARISON BETWEEN THE LOCALIZATION AND SECOND-ORDER WORK CRITERIA FOR PLANE-STRAIN BI-EXTENSIONS

The results presented above permit one to interpret both criteria for a given loading path. More precisely, because of some analogies of the elastic–damage models under consideration with non-associative elastoplastic ones, we look for one loading path (at least) for which the critical value of damage d_ℓ relative to the localization criterion is lower than this corresponding to the second-order work criterion (d_s).

The first problem we deal with is that of plane-strain proportional bi-extension of a square domain Ω, such that $\beta = \dot{U}_y / \dot{U}_x = U_y / U_x$ is constant during all the loading-path. The mechanical state ($\boldsymbol{\varepsilon}$, d) obtained is homogeneous.

Two "second-order work curves" are presented on Figs 3 and 4 (SOW1 and SOW2). The first of them (SOW1) gives the value of damage when the second-order work criterion

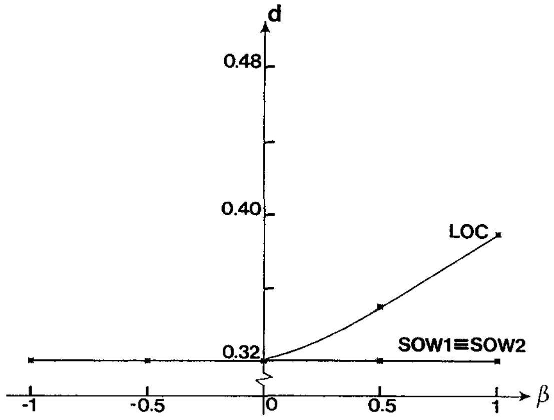

Fig. 3. Model 1; critical damage versus β (proportional loading path): LOC: damage at localization (d_ℓ); SOW_1: damage verifying second-order work criterion (d_s); and SOW_2: damage at loss of positiveness of second-order work for the considered loading path.

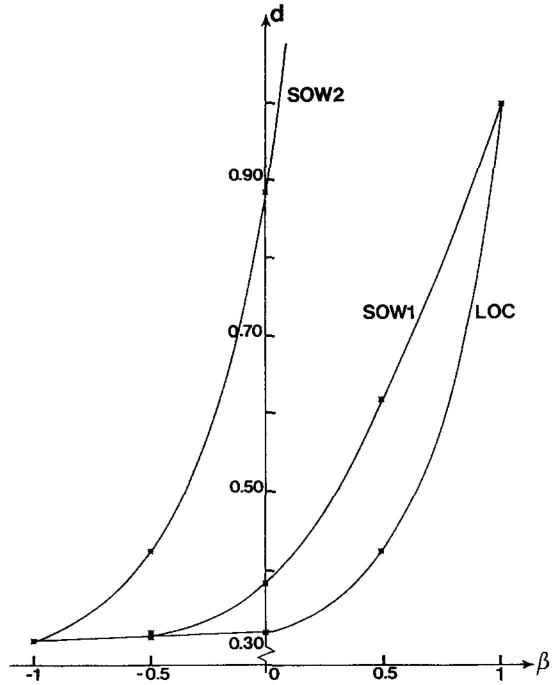

Fig. 4. Model 2 ; critical damage versus β (proportional loading path) : LOC : damage at localization (d_ℓ) ; SOW_1 : damage verifying second-order work criterion (d_s) ; and SOW_2 : damage at loss of positiveness of second-order work for the considered loading path.

is satisfied. This criterion is a sufficient, but not a necessary one, for the loss of positiveness of second-order work. That means that for a given loading path, mechanical states for which the second-order work is positive may exist even though the second-order work criterion is satisfied. The second curve (SOW2) gives the damage value which actually makes zero second-order work for a given value of β.

Figure 4 clearly shows for Model 2 that localization may occur before loss of positiveness of second-order work, i.e. in hardening phase ($\dot{\sigma} : \dot{\varepsilon} > 0$). On the other hand, one can see from Fig. 3 that $d_\ell \geqslant d_s$ for Model 1, i.e. localization always occurs in softening phase.

In fact, some other results based on finite element computations coupled with a numerical version of the localization criterion seem to give prominence to an analogous conclusion, i.e. localization cannot occur in the hardening phase. To give more generality to this result established for Model 1, we then propose a new expression of the localization criterion. This is inspired by the recent works by Ottosen and Runesson (1991) who consider the localization through spectral analysis.

5. SPECTRAL PROPERTIES OF THE LOCALIZATION CRITERION

Denoting by **Q** the characteristic tangent stiffness modulus tensor (or acoustic tensor), i.e.:

$$Q_{ik} = n_j L_{ijkl} n_l, \tag{22}$$

the localization condition (see Section 3.1) can be rewritten as follows:

$$Q_{ik} g_k = 0. \tag{23}$$

Non-trivial solutions of (23) are possible only when **Q** is singular; it is then natural to study the spectral properties of **Q**. More precisely, following Ottosen *et al.* (1991), we consider the (right) eigenvalue problem:

$$Q_{ik}y_k^{(i)} = \lambda^{(i)}Q_{ik}^{ed}y_k^{(i)} \quad i = 1, 2, 3, \tag{24}$$

where:

$$Q_{ik}^{ed} = (1-d)n_jC_{ijkl}^0n_l. \tag{25}$$

Since $\mathbf{Q}^{ed}$ is symmetrical and positive definite (like $\mathbf{C}^0$), it possesses the symmetrical and positive definite inverse $\mathbf{P}^{ed}$. Assuming $d \neq 1$, the eigenvalue problem (24) can then be rewritten:

$$B_{ik}y_k^{(i)} = \lambda^{(i)}y_i^{(i)}, \tag{26}$$

where, due to the particular definition of $\mathbf{L}$ for Model 1 (see Section 2), $\mathbf{B}$ is:

$$B_{ik} = \delta_{ik} - \frac{1}{k_0\alpha}P_{il}^{ed}(n_jC_{ljmn}^0\varepsilon_{mn})(C_{kpqr}^0\varepsilon_{qr}n_p). \tag{27}$$

It can then be observed that 1 is an eigenvalue with a multiplicity of two. We then have for the remaining eigenvalue λ:

$$B_{ii} = 2+\lambda, \tag{28}$$

where B_{ii} can be specified using eqn (27), i.e.:

$$B_{ii} = 3 - \frac{1}{k_0\alpha}P_{il}^{ed}(n_jC_{ljmn}^0\varepsilon_{mn})(C_{ipqr}^0\varepsilon_{qr}n_p). \tag{29}$$

It finally appears that there exists only one possibility for a non-trivial solution of the localization criterion (see eqn (15) section 3.1), namely that $\lambda = 0$. Then, expressing $\boldsymbol{\varepsilon}$ in a coordinate system colinear with its principal directions and splitting it into deviatoric and volumetric parts, this condition ($\lambda = 0$) gives:

$$\frac{(1-d)k_0\alpha}{4G} = (\bar{\varepsilon}_1^2+r\bar{\varepsilon}_1)n_1^2+(\bar{\varepsilon}_2^2+r\bar{\varepsilon}_2)n_2^2+(\bar{\varepsilon}_3^2+r\bar{\varepsilon}_3)n_3^2-\psi(\bar{\varepsilon}_1n_1^2+\bar{\varepsilon}_2n_2^2+\bar{\varepsilon}_3n_3^2)-k, \tag{30}$$

where,

$$G = \frac{E}{2(1+\nu)}, \quad \psi = \frac{1}{2(1-\nu)}, \quad r = \frac{1+\nu}{3(1-\nu)}\varepsilon_{kk}, \quad k = -\frac{(1+\nu)^2}{18(1-\nu)(1-2\nu)}\varepsilon_{kk}^2,$$

and

$$\bar{\varepsilon}_i = \varepsilon_i - \tfrac{1}{3}\varepsilon_{kk}.$$

The relation between $\boldsymbol{\varepsilon}$ and d is simply obtained from the damage loading condition, namely $g(F_d; d) = 0$ (see Section 2).

The critical damage value at localization d_ℓ is then defined as the minimum value of d verifying (30), i.e. with respect to a variation of the localization band direction n_i for a given state:

$$d_\ell = \min d(n_i). \tag{31}$$

Equation (30) is fully analogous to that obtained by Ottosen and Runesson (1991) and defining the critical hardening modulus at localization for a general class of plasticity models which shows once more the analogies between elastic damage models and non-associative elastoplastic ones. The results established by these authors can be then used to specify the critical damage value at localization (d_ℓ) which appears to be function of α and ν only and of the ratio between strain components, i.e.

$$d_\ell \equiv d_\ell(\alpha, \nu, \eta, \chi), \tag{32}$$

where,

$$\eta = \frac{\varepsilon_2}{\varepsilon_1}, \chi = \frac{\varepsilon_3}{\varepsilon_1} (\varepsilon_1 \neq 0). \tag{33}$$

Remark

The procedure leading to d_ℓ makes it necessary to distinguish various strain paths in relation to η and χ. The analogous procedure in plasticity is detailed by Ottosen and Runesson (1991). It then appears that for each particular strain path, a very complex explicit form for d_ℓ is obtained; this is not performed here. To determine whether the localization can occur before loss of positiveness of second-order work, it is sufficient to determine $d_\ell^{\min}$ (see below) and not particular d_ℓ. However, as an example, the procedure leading to both d_ℓ and $d_\ell^{\min}$ is given in Appendix 3 for plane-strain bi-extensions (see also Section 4).

To compare d_ℓ with the critical damage value at loss of positiveness of second-order work d_s (see Section 3.2) we then look for the lower bound $d_\ell^{\min}$ of d_ℓ. The purpose is thus to find this critical value $d_\ell^{\min}$ which is defined as the minimum value of d with respect to ν, η and χ, i.e.

$$d_\ell^{\min}(\alpha) = \min d_\ell(\alpha, \nu, \eta, \chi). \tag{34}$$

An example of this minimization procedure is given in Appendix 3. It finally appears that, whatever mechanical state (characterized by η and χ) considered, $d_\ell^{\min}$ is defined by:

$$d_\ell^{\min}(\alpha) = \frac{\alpha - 1}{3\alpha}, \tag{35}$$

which is equal to d_s. Thus, despite some analogy with non-associative elastoplastic models, this elastic–damage model (Model 1) does not allow localization occurring in hardening phase. From this point of view, it presents an obvious similarity with associative elastoplasticity.

6. CONCLUSION

Two local criteria for loss of uniqueness of velocity gradients for the local rate problem have been established for "d-associative" isotropic elastic–damage models. For each of two models, the critical damage value corresponding to loss of positiveness of the second-order work was explicitly calculated by a constrained minimization process. The critical damage value at localization has then been calculated for specific loading paths (plane-strain bi-extensions): these results have been sufficient to confirm the analogy to the non-associative plasticity of the one of two models (Model 2), the localization being allowed to occur in hardening phase. On the other hand, for the other elastic–damage model (Model 1), only a spectral analysis of the localization problem permitted the determination of the critical damage value at localization for the arbitrary loading path. It then appeared that this model does not allow the localization occurring in hardening phase, the same being true for the associative plasticity.

These results show in the first place that the results established in elastoplasticity for the loss of uniqueness of velocity gradients, for the local rate problem *cannot be simply* *extended to damage models*: the existing analogies between "*d*-associative" and non-associative elastoplastic models (which arise when no particular assumptions are made on the evolution law of the inelastic part of the strain rate tensor due to damage growth) should not overshadow the differences occurring when considering the localization mechanisms.

Otherwise, these results confirm the unfitness of Model 1 to describe the mechanical state for the onset of localization in brittle rock-like materials for which numerous experimental data clearly show that the localization can occur in hardening phase (Santarelli, 1989). Improved models, using for example a tensorial damage internal variable (Dragon *et al.*, 1993) describing the oriented character of microcrack-sets in brittle rock-like materials would be able to describe in a much more realistic manner the localization events in such materials. When considering a scalar damage variable, an alternative solution can be found by considering higher-order displacement gradients. Thus, following the work of Zbib and Aifantis (1989) on the gradient-dependent theory of plasticity we can view the introduction of a gradient-dependent damage limit in Model 1, these higher order gradients describing, in some manner, the macroscopic manifestation of the inhomogeneous evolution of the microstructure of the material. Besides the fact that the aforesaid authors showed, in the elastoplastic case, that these higher-order gradients influence the critical mechanical state for the onset of localization without altering the orientation of localization bands, this approach has the ability to give an estimation of the thickness of the localization band, the latter being impossible using classical models.

Acknowledgement—The authors are greatly indebted to Professor A. Dragon for his inspiration and comments during the preparation of this paper.

REFERENCES

Benallal, A., Billardon, R. and Geymonat, G. (1989). Conditions de bifurcation à l'intérieur et aux frontières pour une classe de matériaux non-standards. *C. R. Acad. Sci. Paris* 308, série **II**, 893–898.

Borré, G. and Maier, G. (1989). On linear versus non-linear flow rules in strain localization analysis. *Meccanica* **24**, 36–41.

Chambon, R. (1986). Bifurcation par localisation en bande de cisaillement, une approche avec des lois incrémentalement non linéaires. *J. Méca. Th. Appl.* **5**, 277–298.

Désoyer, T., Charlez, P. and Dragon, A. (1990). Approche par éléments finis d'un problème d'endommagement des roches. Proc. Strucome Congress, Edn. Dataid, Paris, 381–388.

Dragon, A., Charlez, Ph., Pham. D. and Shao, J. F. (1993). A model of anisotropic damage by (micro)crack growth. In *Assessment and Prevention of Failure Phenomena in Rock Engineering* (Edited by A. G. Pasamehmetoglu, T. Kawamoto, B. N. Whittaker and O. Aydan), pp. 71–78. Balkema, Rotterdam.

Hadamard, J. (1903). *Leçons sur la Propagation des Ondes et les Equations de l'Hydrodynamique*, Chapter 6. Edn. Hermann, Paris.

Hill, R. (1958). A general theory of uniqueness and stability in elastic–plastic solids. *J. Mech. Phys. Solids* **6**, 236–249.

Hill, R. (1962). Acceleration waves in solids. *J. Mech. Phys. Solids* **10**, 1–16.

Hueckel, T. and Maier, G. (1977). Incremental boundary value problems in the presence of coupling of elastic and plastic deformations: a rock mechanics oriented theory. *Int. J. Solids Structures* **13**, 1–15.

Lemaître, J. and Chaboche, J. L. (1978). Aspect phénoménologique de la rupture par endommagement. *J. Méca. Appl.* **2**, 317–366.

Mandel, J. (1966). Conditions de stabilité et postulat de Drucker. In *Rheology and Soils Mechanics* (Edited by J. Kravtchenko and P. M. Sirieys), pp. 58. Springer, Berlin.

Ottosen, N. S. and Runesson, K. (1991). Properties of discontinuous bifurcation solutions in elastoplasticity. *Int. J. Solids Struct.* **27**, 401–421.

Raniecki, B. and Bruhns, O. T. (1981). Bounds to bifurcation stresses in solids with non-associated plastic flow laws at finite deformations. *Bull. Acad. Polon. Sci. Ser. Tech.* **XXVIII**, 391.

Rice, J. R. (1976). The localization of plastic deformation. In *Theoretical and Applied Mechanics* (Edited by W. T. Koiter), pp. 207. North-Holland, Amsterdam.

Rice, J. R. and Rudnicki, J. W. (1979). A note on some features of the theory of localization of deformation. *Int. J. Solids. Struct.* **16**, 597–605.

Santarelli, F. J. (1989). Etude expérimentale de la bifurcation dans les roches. *Rev. Fr. Geotech.* **50**, 61–70.

Zbib, H. M. and Aifantis, E. C. (1989). A gradient-dependent flow theory of plasticity: Application to metal and soil instabilities. *Appl. Mech. Rev.* **42**, 295–304.

APPENDIX 1. EXPRESSION OF THE COEFFICIENTS $A_i (i = 1\text{–}5)$ APPEARING IN THE LOCALIZATION CRITERION FOR BOTH OF THE ELASTIC–DAMAGE MODELS IN THE PLANE STRAIN CASE

Expressing the damage branch $\mathbf{H}$ of the tangent matrix in $(\mathbf{x}_1, \mathbf{x}_2)$ system of coordinates such that $\mathbf{n} = (0, 1)$, the localization criterion [see Section 3.1, eqn (15)] is rewritten:

$$\det(H_{i2k2}) = 0. \tag{A1}$$

For both of the elastic–damage models considered here, (A1) is equivalent to:

$$A_1 \cos^4\theta + A_2 \cos^2\theta + A_3 \sin\theta\cos^3\theta + A_4 \sin\theta\cos\theta + A_5 = 0 \tag{A2}$$

with $\cos\theta = \mathbf{n}\cdot\mathbf{y}$, where the coefficients $A_i (i = 1\text{–}5)$ are functions of material constants $(E, \nu; k_0, \alpha)$ and of ε and d. We give below the expression of these coefficients A_i for each of the models in the plane strain case.

Model 1

$$\begin{aligned}
A_1 &= -E[(\varepsilon_{yy}-\varepsilon_{xx})^2-4\varepsilon_{xy}^2],\\
A_2 &= 2E[\varepsilon_{yy}(\varepsilon_{yy}-\varepsilon_{xx})-2\varepsilon_{xy}^2],\\
A_3 &= 4E\varepsilon_{xy}(\varepsilon_{yy}-\varepsilon_{xx}),\\
A_4 &= -4E\varepsilon_{xy}\varepsilon_{yy},\\
A_5 &= 2E(1-\nu)\varepsilon_{xy}^2+[E[(1-\nu)\varepsilon_{xx}+\nu\varepsilon_{yy}]^2/(1-2\nu)]-k_0\alpha(1-d)(1-\nu)(1+\nu).
\end{aligned} \tag{A3}$$

Model 2

$$\begin{aligned}
A_1 &= 4G[G(1-d)+3K][4\varepsilon_{xy}^2-(\varepsilon_{yy}-\varepsilon_{xx})^2],\\
A_2 &= 2G[6K\varepsilon_{xy}^2+2[2G(1-d)+3K]\varepsilon_{yy}^2-4[G(1-d)+3K](2\varepsilon_{xy}^2+\varepsilon_{xx}\varepsilon_{yy})],\\
A_3 &= 16G[G(1-d)+3K]\varepsilon_{xy}(\varepsilon_{yy}-\varepsilon_{xx}),\\
A_4 &= (8G/3)\{[4G(1-d)+9K]\varepsilon_{xx}-[10G(1-d)+9K]\varepsilon_{yy}\}\varepsilon_{xy},\\
A_5 &= (16G^2/3)(1-d)(\varepsilon_{xy}^2-\varepsilon_{yy}\varepsilon_{xx})+(4G^2/3)(1-d)(\varepsilon_{yy}^2+12\varepsilon_{xy}^2)+12GK\varepsilon_{xy}^2\\
&\quad -(1-d)k_0\alpha[4G(1-d)+3K],
\end{aligned} \tag{A4}$$

where:

$$G = \frac{E}{2(1+\nu)}; \quad K = \frac{E}{3(1-2\nu)}.$$

APPENDIX 2. THREE-DIMENSIONAL SECOND-ORDER WORK CRITERION FOR BOTH ELASTIC–DAMAGE MODELS

Let us recall that for both elastic–damage models, the constitutive law can be written in the unique general manner:

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{C}(d):\boldsymbol{\varepsilon}\; F_d = -\frac{1}{2}\boldsymbol{\varepsilon}:\mathbf{C}':\boldsymbol{\varepsilon},\\ \dot{d} = \dfrac{(-\mathbf{C}':\boldsymbol{\varepsilon}:\dot{\boldsymbol{\varepsilon}})^+}{k_0\alpha}, \end{cases} \tag{A5}$$

from which we have:

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}(d):\dot{\boldsymbol{\varepsilon}}+\dot{d}\mathbf{C}':\boldsymbol{\varepsilon}. \tag{A6}$$

Thus, second-order work criterion becomes:

$$W(\dot{\boldsymbol{\varepsilon}}) = \tfrac{1}{2}\dot{\boldsymbol{\sigma}}:\dot{\boldsymbol{\varepsilon}} = \tfrac{1}{2}[\mathbf{C}(d):\dot{\boldsymbol{\varepsilon}}:\dot{\boldsymbol{\varepsilon}}+\dot{d}\mathbf{C}':\boldsymbol{\varepsilon}:\dot{\boldsymbol{\varepsilon}}] = 0. \tag{A7}$$

Accounting for the definite-positiveness of $\mathbf{C}(d)$, which assures that $W(\dot{\boldsymbol{\varepsilon}})$ is always strictly positive in purely elastic phase and the fact that only the sign of $W(\dot{\boldsymbol{\varepsilon}})$ is important for our purpose, we may suppose $\dot{d} = 1$.† Thus, the second-order work criterion can be rewritten as:

$$\begin{cases} W(\dot{\boldsymbol{\varepsilon}}) = \frac{1}{2}[\mathbf{C}(d):\dot{\boldsymbol{\varepsilon}}:\dot{\boldsymbol{\varepsilon}}+\mathbf{C}':\boldsymbol{\varepsilon}:\dot{\boldsymbol{\varepsilon}}] = 0,\\ \text{with } \mathbf{C}':\boldsymbol{\varepsilon}:\dot{\boldsymbol{\varepsilon}}+k_0\alpha = 0, \end{cases} \tag{A8}$$

the second equation (A8) signifying a damage loading ($\dot{g} = 0$).

$W(\dot{\boldsymbol{\varepsilon}})$ is strictly convex. Thus, the study of second-order work criterion is equivalent to the study of the sign of the minimum of $W(\dot{\boldsymbol{\varepsilon}})$ with the constraint of damage loading, i.e.

$$\mathrm{Sgn}\left\{\min_{\dot{\boldsymbol{\varepsilon}}} \tfrac{1}{2}[\mathbf{C}(d):\dot{\boldsymbol{\varepsilon}}:\dot{\boldsymbol{\varepsilon}}+\mathbf{C}':\boldsymbol{\varepsilon}:\dot{\boldsymbol{\varepsilon}}] \quad \text{with } \mathbf{C}':\boldsymbol{\varepsilon}:\dot{\boldsymbol{\varepsilon}}+k_0\alpha = 0\right\}. \tag{A9}$$

This equality-constrained minimization problem equation reduces to the unconstrained minimization of the Lagrangean function:

† All mathematical derivations presented are fully independent of this assumption ($\dot{d} = 1$). It is just to simplify the mathematical writing that this choice was made.

$$W^*(\dot{\varepsilon},\omega) = \tfrac{1}{2}(\mathbf{C}(d):\dot{\varepsilon}:\dot{\varepsilon}+\mathbf{C}':\varepsilon:\dot{\varepsilon})+\omega(\mathbf{C}':\varepsilon:\dot{\varepsilon}+k_0\alpha), \tag{A10}$$

where ω is a Lagrangean multiplier. The derivatives of W^* with respect to all variables must vanish, i.e.

$$\begin{cases} \dfrac{\partial W^*}{\partial \dot{\varepsilon}} = \mathbf{C}(d):\dot{\varepsilon}+\tfrac{1}{2}\mathbf{C}':\varepsilon+\omega\mathbf{C}':\varepsilon = 0, \\ \dfrac{\partial W^*}{\partial \omega} = \mathbf{C}':\varepsilon:\dot{\varepsilon}+k_0\alpha = 0. \end{cases} \tag{A11}$$

The solutions of these equations are:

$$\begin{cases} \dot{\varepsilon}^0 = -\left(\dfrac{1+2\omega_0}{2}\right)\mathbf{D}(d):\mathbf{C}':\varepsilon \quad \text{where } [\mathbf{D}(d) \equiv \mathbf{C}^{-1}(d)], \\ \omega_0 = \dfrac{k_0\alpha}{\mathbf{C}':\varepsilon:\mathbf{D}(d):\mathbf{C}':\varepsilon} - \dfrac{1}{2}. \end{cases} \tag{A12}$$

The minimum work $W_0 = W(\dot{\varepsilon}^0)$ must vanish when the second-order work criterion is satisfied. Thus, the criterion is written:

$$\mathbf{C}':\varepsilon:\mathbf{D}(d):\mathbf{C}':\varepsilon = k_0\alpha, \tag{A13}$$

which finally gives:

$$d = \frac{\alpha-1}{3\alpha}. \tag{A14}$$

APPENDIX 3. AN EXAMPLE OF THE LOWER BOUND OF $d_\ell(d_\ell^{\min})$: THE PLANE-STRAIN BI-EXTENSIONS

Following the results established by Ottosen *et al.* (1991), the determination of the critical damage value at localization d_ℓ makes it necessary to distinguish diverse strain configurations. In the peculiar case of plane-strain bi-extensions, η and χ (see Section 4.2) are such that (assuming $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \varepsilon_2$):

$$\chi = 0; \quad \eta \in [0,1], \tag{A15}$$

d_ℓ is then given by:

$$d_\ell(\alpha,\nu,\eta) = \frac{(\alpha-1)[(1-\nu)^2+\eta\nu(2-2\nu+\eta\nu)]+\alpha\eta^2(1-2\nu)}{3\alpha[(1-\nu)^2+\eta\nu(2-2\nu+\eta\nu)]+\alpha\eta^2(1-2\nu)}. \tag{A16}$$

Looking for the minimum of $d_\ell(\alpha,\ \nu,\ \eta)$ with respect to η, we can then observe that:

$$\mathrm{Sgn}\,(d_{\ell,\eta}) = \mathrm{Sgn}\,[2\alpha(2\alpha+1)(\nu-1)(2\nu-1)\eta(1-\nu+\eta\nu)], \tag{A17}$$

where

$$d_{\ell,\eta} = \frac{\partial d_\ell}{\partial \eta}.$$

From $\alpha \geqslant 1$ (see Section 2), $\nu \in [0, \tfrac{1}{2})$, $\eta \in [0, 1]$, $d_{\ell,\eta}$ is always positive and d_ℓ is minimum when $\eta = 0$, i.e.

$$d_\ell^{\min} = d_\ell(\alpha,\nu,0) = \frac{\alpha-1}{3\alpha}, \tag{A18}$$

which finally confirms that localization cannot occur in the hardening phase for plane-strain bi-extensions (see also Section 4.1).